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NOTE

Completeness of Trigonometric System with Integer Indices $\{e^{inx};\, x \in \mathfrak{R}\}$

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the trigonometric systems with integer indices; $\{e^{inx};\, x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ or $\{e^{inx};\, x \in \mathfrak{R}\}_{n=1}^{\infty}$ in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$. If there exists a support Λ of the measure μ which is a wandering set, that is, $\Lambda + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ are mutually disjoint for different k's, then the linear span of our trigonometric system $\{e^{inx};\, x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is dense in $L^{\alpha}(\mu, \mathfrak{R})$ $\alpha \geqslant 1$. The converse statement is also true.

1. INTRODUCTION

Let μ be a finite positive Borel measure. In the case the support of μ is narrowly bounded, for example, *supp* $\mu = [-\pi, \pi)$, it can be shown that $\{e^{inx};\, x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is complete in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$, since if $f \in L^{\beta}(\mu, \mathfrak{R})$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $\int_{-\infty}^{\infty} f(x)\, e^{inx} \mu(dx) = \int_{-\pi}^{\pi} f(x)\, e^{inx} \mu(dx) = 0$, $n = 0, \pm 1, \pm 2, \ldots$ implies that $f(x)$ is a null function (a.e. μ) and in view of Hahn–Banach theorem, this is equivalent to the fact that $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$ is equal to the linear span by $\{e^{inx};\, x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$. On the other hand, if the width of support of μ is not so narrow, for example, supp $\mu = [-\pi - h, \pi + h]$, $\pi > h > 0$, taking μ to be a Lebesgue measure $\mu(dx) = dx$ in $[-\pi - h, \pi + h]$, and $\mu(dx) = 0$ outside of $[-\pi - h, \pi + h]$ on the real line, it can be proved that $\{e^{inx};\, x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is not complete on $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$ (Young [3, p. 113]). It should be noticed that whether $\{e^{inx};\, x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is

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complete or not depends only on the width of the support of the measure and not on the position of its support in the real line. In fact, we will see from Theorem 1 proved in Section 2 that $\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is complete in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$ for μ whose support is a wandering set in the ergodic theory sense; that is, *supp* $\mu = \bigcup_{k=-\infty}^{\infty} (a_k + 2k\pi, b_k + 2k\pi)$, where $(a_k, b_k) \subseteq [-\pi, \pi)$ are mutually disjoint intervals for each different integers k. In this paper we will give necessary and sufficient conditions ensuring the completeness for the total system $\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$ (Theorem 1), and for the the half system, $\{e^{inx}; x \in \mathfrak{R}\}_{n=1}^{\infty}$ in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$ (Theorem 2). Furthermore Theorem 3 deals with necessary and sufficient conditions such that $e^{i\frac{q}{p}x}$ is contained in the closure set of finite linear combinations of elements from $\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$, when an integer $q \neq 0$ is relatively prime to a certain positive integer $p \geqslant 2$, that is, q and p do not have any common divisor except one. These results can be applied to the sampling theorem for stationary stochastic processes. In connection with this, Lloyd [2] has proved the completeness of $\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ in $L^{2}(\mu, \mathfrak{R})$ by using some ergodic theorems, although our proof will be carried out in a different way.

2. THEOREMS

Let μ be a finite positive Borel measure on the real line $\mathfrak{R}$. We will call a set Λ the support of a measure μ if for any $\delta > 0$, $\mu\{(x-\delta, x+\delta)\} > 0$ for each $x \in \Lambda$. We easily see that Λ is a closed set. Let us consider a decomposition of Λ,

$$\Lambda = \bigcup_{k=-\infty}^{\infty} E_k \tag{1}$$

$$E_k = \Lambda \cap [(2k-1)\pi, (2k+1)\pi), \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2}$$

and consider their translations into $[-\pi, \pi)$:

$$A_k = E_k - 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{3}$$

We will use the following facts in proving the theorems.

LEMMA 1. *$\Lambda + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ are mutually disjoint for different k if and only if A_k, $k = 0, \pm 1, \pm 2, \ldots$, are mutually disjoint for different k.*

THEOREM 1. *Let μ be a finite positive Borel measure on $\mathfrak{R}$. Then* (a), (b), *and* (c) *are equivalent.*

(a) *Λ is the support set of μ such that $\Lambda+2k\pi$, $k=0, \pm 1, \pm 2, \ldots$ are mutually disjoint for different k's.*

(b) *$\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is complete in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$.*

(c) *For an irrational number ξ, $e^{i\xi x} \in cl\ \{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$, where the closure is taken in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$, and $cl\{\}$ means the closed linear hull by elements belonging to the set $\{\}$.*

Proof of (a) $\Rightarrow$ (b). Assume that

$$\int_{-\infty}^{\infty} e^{inx} g(x)\, \mu(dx) = 0, \quad n=0, \pm 1, \pm 2, \ldots \tag{4}$$

for a $g \in L^{\beta}(\mu, \mathfrak{R})$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, $\alpha \geqslant 1$, where β is to be $+\infty$ when $\alpha=1$. We must show that $g(x)=0$ (a.e. μ). We take a function $h(x)$ defined on $[-\pi, \pi)$ such that $h(x)=g(x+2k\pi)$ for $x \in A_k$, $k=0, \pm 1, \pm 2, \ldots$, $h(x)=0$, for $x \notin \bigcup_{k=-\infty}^{\infty} A_k$, where A_k being in (3). We also define a measure ν as $\nu(dx)=\mu(dx+2k\pi)$ for $x \in A_k$, $k=0, \pm 1, \pm 2, \ldots$; $\nu(dx)=0$ for $x \notin \bigcup_{k=-\infty}^{\infty} A_k$. h and ν are well defined on $[-\pi, \pi)$, since A_k are mutually disjoint for $k=0, \pm 1, \pm 2, \ldots$ by virtue of the lemma. We can rewrite (3) as

$$\begin{aligned}\int_{-\infty}^{\infty} e^{inx} g(x)\, \mu(dx) &= \sum_{k=-\infty}^{\infty} \int_{E_k} e^{inx} g(x)\, \mu(dx)\\ &= \sum_{k=-\infty}^{\infty} \int_{A_k} e^{inx} g(x+2\pi k)\, \mu(dx+2\pi k)\\ &= \int_{-\pi}^{\pi} e^{inx} h(x)\, \nu(dx) = 0, \qquad n=0, \pm 1, \pm 2, \ldots.\end{aligned} \tag{5}$$

However, the uniqueness of Fourier Stieltjes coefficients defined on $[-\pi, \pi)$ implies that $h(x)=0$ (a.e. ν), which also implies the required result $g(x)=0$, a.e. μ. ∎

Proof of (b) $\Rightarrow$ (c). Trivial. ∎

Proof of (c) $\Rightarrow$ (a). We assume that we can find integers k and l such that $A_k \cap A_l \neq \phi$ and we will show that this assumption implies a contradiction. From the assumption there exist polynomials $p_N(x)=\sum_{|k| \leqslant N} a_k e^{ikx}$ such that $\int_{-\infty}^{\infty} |e^{i\xi x}-p_N(x)|^{\alpha}\, \mu(dx) < \varepsilon_N$, $\varepsilon_N \to 0$ as $N \to \infty$. Hence for $k \neq l$, $\int_{E_k} |e^{i\xi x}-p_N(x)|^{\alpha}\, \mu(dx)+\int_{E_l} |e^{i\xi x}-p_N(x)|^{\alpha}\, \mu(dx) < \varepsilon_N$. Define a positive measure $\omega(E)=\inf\{\mu(E+2\pi k), \mu(E+2\pi l)\}$, where E is an arbitrary measurable set in $[-\pi, \pi)$. Then we have

$$
\begin{aligned}
&|e^{i\xi 2\pi k}-e^{i\xi 2\pi l}|^{\alpha}\,\omega(A_k\cap A_l)\\
&\quad\leqslant 2^{\alpha-1}\int_{A_k\cap A_l}|e^{i\xi(x+2\pi k)}-p_N(x)|^{\alpha}\,\omega(dx)\\
&\qquad+2^{\alpha-1}\int_{A_k\cap A_l}|e^{i\xi(x+2\pi l)}-p_N(x)|^{\alpha}\,\omega(dx)\\
&\quad\leqslant 2^{\alpha-1}\int_{A_k}|e^{i\xi(2\pi k+x)}-p_N(x)|^{\alpha}\,\mu(dx+2\pi k)\\
&\qquad+2^{\alpha-1}\int_{A_l}|e^{i\xi(2\pi l+x)}-p_N(x)|^{\alpha}\,\mu(dx+2\pi l)\\
&\quad=2^{\alpha-1}\int_{E_k}|e^{i\xi x}-p_N(x)|^{\alpha}\,\mu(dx)+2^{\alpha-1}\int_{E_l}|e^{i\xi x}-p_N(x)|^{\alpha}\,\mu(dx)\\
&\quad\leqslant 2^{\alpha-1}\varepsilon_N. \qquad (6)
\end{aligned}
$$

In other words, if we take $p_N(x)$ properly and $N\to\infty$, we obtain $\omega(A_k\cap A_l)=0$. However, $A_k\cap A_l$ was a set of support points both of $\mu(dx+2\pi k)$ and of $\mu(dx+2\pi l)$ so of $\omega(dx)$ or we should have $\omega(A_k\cap A_l)>0$. Hence we deduce from this contradiction that we must have $A_k\cap A_l=\phi$. ∎

THEOREM 2. *Let μ be a finite positive Borel measure on $\mathfrak{R}$. The following* (a), (b), (c) *are equivalent.*

(a) *Λ is a support of μ such that $\Lambda+2k\pi$, $k=0,\pm1,\pm2,\ldots$ are mutually disjoint for different k and*

$$\int_{-\pi}^{\pi}\log v'(x)\,dx=-\infty, \qquad (7)$$

where $v'(x)$ being the derivative of the absolutely continuous part of $v(dx)=\mu(dx+2k\pi)$ for $x\in A_k$, $k=0,\pm1,\pm2,\ldots$

(b) *The linear span by $\{e^{inx};x\in\mathfrak{R}\}_{n=1}^{\infty}$ is dense in $L^{\alpha}(\mu,\mathfrak{R})$, $\alpha\geqslant1$.*

(c) *For an irrational number ξ, $e^{i\xi x}\in cl\{e^{inx};x\in\mathfrak{R}\}_{n=1}^{\infty}$ and $1\in cl\{e^{inx};x\in\mathfrak{R}\}_{n=1}^{\infty}$.*

Proof of (a) ⇒ (b). Suppose that there exists $g(x)\in L^{\alpha}(\mu,\mathfrak{R}))$, $\alpha\geqslant1$ such that

$$\int_{-\infty}^{\infty}e^{inx}g(x)\,\mu(dx)=0, \qquad (8)$$

for $n = 1, 2, 3, \ldots$. Applying the same notation and the same reasoning as in the proof of Theorem 1, the first assumption (a) implies that

$$\int_{-\infty}^{\infty} e^{inx} g(x)\, \mu(dx) = \int_{-\pi}^{\pi} e^{inx} h(x)\, \nu(dx) \tag{9}$$

for $n = 1, 2, 3, \ldots$. The second assumption (a) implies that the linear span of $\{e^{inx}; x \in \mathfrak{R}\}_{n=1}^{\infty}$ is dense in $L^{\alpha}(\nu, [-\pi, \pi))$, $\alpha \geqslant 1$ by Szegö's theorem (Akhiezer [1, p. 262]). Hence from (8) and (9), $h(x) = 0$ (a.e. ν) and $g(x) = 0$ (a.e. μ). ▮

Proof of (b) ⇒ (c). Trivial. ▮

Proof of (c) ⇒ (a). The condition $1 \in cl\{e^{inx}; x \in \mathfrak{R}\}_{n=1}^{\infty}$ means that $cl\{e^{inx}; x \in \mathfrak{R}\}_{n=1}^{\infty} = cl\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$. Hence the first assumption of (c) implies that $e^{i\xi x} \in cl\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ which is exactly (c) of Theorem 1 and we get the required result. ▮

COROLLARY 1. *If there exists a support set Λ of a finite positive Borel measure μ such that $\Lambda + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ being mutually disjoint for different k and the width of Λ is less than 2π, then the linear span by $\{e^{inx}\}_{n=1}^{\infty}$ is dense in $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$.*

Proof. The width of Λ equals

$$|\Lambda| = \sum_{k=-\infty}^{\infty} |A_k| \tag{10}$$

because A_k are disjoint. If $|\Lambda| < 2\pi$, then $\nu(I) = 0$ for an interval of positive Lebesgue measure contained in $[-\pi, \pi)$ so it yields that

$$\log \nu'(x) \notin L^1(dx, [-\pi, \pi)). \tag{11}$$

From (a) of Theorem 2, we have the desired result. ▮

We have one more generalization. We will define the set $\tilde{A}_j = \bigcup_{l = j (\mathrm{mod}\, p)} A_l$.

LEMMA 2. *$\Lambda + 2j\pi$, $j = 0, 1, \ldots, p-1$ are mutually disjoint for different integers j if and only if $\tilde{A}_j$ are mutually disjoint for different j, $j = 0, 1, \ldots, p-1$.*

THEOREM 3. *Let μ be a finite positive Borel measure on $\mathfrak{R}$ and p be a positive integer $\geqslant 2$. Then* (a), (b), *and* (c) *are equivalent.*

(a) *There exists a support set Λ of μ such that $\Lambda + 2j\pi$ are mutually disjoint for different integers j, where $j = 0, 1, 2, \ldots, p-1$.*

(b) *For any integer* q, $e^{i(q/p)x} \in cl\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$, *the closure being taken in* $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$.

(c) *For an integer* p *relatively prime to* q, $e^{i(q/p)x} \in cl\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$, *the closure being taken in* $L^{\alpha}(\mu, \mathfrak{R})$, $\alpha \geqslant 1$

Proof of (a) $\Rightarrow$ (b). From the lemma, $\tilde{A}_j = \bigcup_{l=j(\mathrm{mod}\, p)} A_l$ are mutually disjoint for different j, $j = 0, 1, 2, ..., p-1$, and let $\lambda(dx) = \sum_{k=-\infty}^{\infty} \mu(dx + 2\pi k)$, x in $[-\pi, \pi)$. For any polynomial $p_N(x) = \sum_{|k| \leqslant N} a_k e^{ikx}$, we have

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} |e^{i\frac{q}{p}x} - p_N(x)|^{\alpha}\, \mu(dx)\right)^{\frac{1}{\alpha}} \\
&\quad = \left(\sum_{k=-\infty}^{\infty} \int_{E_k} |e^{i\frac{q}{p}x} - p_N(x)|^{\alpha}\, \mu(dx)\right)^{\frac{1}{\alpha}} \\
&\quad = \left(\sum_{k=-\infty}^{\infty} \int_{A_k} |e^{i\frac{q}{p}(x+2k\pi)} - p_N(x)|^{\alpha}\, \mu(dx+2k\pi)\right)^{\frac{1}{\alpha}} \\
&\quad = \left(\sum_{j=0}^{p-1} \sum_{k=j(\mathrm{mod}\, p)} \int_{A_k} |e^{i\frac{q}{p}(x+2k\pi)} - p_N(x)|^{\alpha}\, \mu(dx+2k\pi)\right)^{\frac{1}{\alpha}} \\
&\quad \leqslant \sum_{j=0}^{p-1} \left(\int_{\tilde{A}_j} |e^{i\frac{q}{p}(x+2j\pi)} - p_N(x)|^{\alpha}\, \lambda(dx)\right)^{\frac{1}{\alpha}} \qquad (12)
\end{aligned}$$

The last term can be less than arbitrarily small positive numbers by choosing $p_N(x)$ properly because $\{e^{inx}; x \in \mathfrak{R}\}_{n=-\infty}^{\infty}$ is known to be complete in $L^{\alpha}(\lambda, [-\pi, \pi))$ with a finite positive measure λ. ∎

Proof of (b) $\Rightarrow$ (c). Trivial. ∎

Proof of (c) $\Rightarrow$ (a). Let $\tilde{A}_j \cap \tilde{A}_k \neq \phi$. Then there exist integers n and m such that $A_n \cap A_m \neq \phi$, where $n = j(\mathrm{mod}\, p)$ and $m = k(\mathrm{mod}\, p)$. Then we have for $\omega(E) = \inf\{\mu(E + 2\pi n), \mu(E + 2\pi m)\}$, where E is an arbitrary measurable set in $[-\pi, \pi)$,

$$\begin{aligned}
&|e^{i\frac{q}{p}2\pi j} - e^{i\frac{q}{p}2\pi k}|^{\alpha}\, \omega(A_n \cap A_m) \\
&\quad = \left(\int_{A_n \cap A_m} |e^{i\frac{q}{p}(2\pi j + x)} - e^{i\frac{q}{p}(2\pi k + x)}|^{\alpha}\, \omega(dx)\right) \\
&\quad \leqslant 2^{\alpha-1}\left(\int_{A_n} |e^{i\frac{q}{p}(2\pi j + x)} - p_N(x)|^{\alpha}\, \omega(dx)\right) \\
&\qquad + 2^{\alpha-1}\left(\int_{A_m} |e^{i\frac{q}{p}(2\pi k + x)} - p_N(x)|^{\alpha}\, \omega(dx)\right)
\end{aligned}$$

$$\leqslant 2^{\alpha-1}\left(\int_{-\pi}^{\pi} |e^{i\frac{q}{p}(2\pi k+x)} - p_N(x)|^\alpha \mu(dx+2\pi m)\right)$$

$$+2^{\alpha-1}\left(\int_{-\pi}^{\pi} |e^{i\frac{q}{p}(2\pi n+x)} - p_N(x)|^\alpha \mu(dx+2\pi m)\right)$$

$$\leqslant 2^{\alpha-1}\int_{-\infty}^{\infty} |e^{i\frac{q}{p}x} - p_N(x)|^\alpha \mu(dx) \tag{13}$$

which can be less than any small number by choosing $p_N(x)$ properly. Hence we should have $\omega(A_n \cap A_m) = 0$. $A_n \cap A_m$ was a set of support points both of $\mu(dx+2\pi n)$ and of $\mu(dx+2\pi m)$ so of $\omega(dx)$. In other words we should have $\omega(A_k \cap A_l) > 0$. Hence we deduce from this contradiction that we must have $\tilde{A}_j \cap \tilde{A}_k = \phi$. ∎

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